

# Remarks on Leibniz algebras <sup>\*</sup>

Yunhe Sheng

Department of Mathematics, Jilin University, Changchun 130012, China  
email: shengyh@jlu.edu.cn

Zhangju Liu

Department of Mathematics, Peking University, Beijing 100871, China  
email: liuzj@pku.edu.cn

## Abstract

In this paper, first we construct a Lie 2-algebra associated to every Leibniz algebra via the skew-symmetrization. Furthermore, we introduce the notion of the naive representation for a Leibniz algebra in order to realize the abstract operations as a concrete linear operation. At last, we study some properties of naive cohomologies.

## 1 The skew-symmetrization of a Leibniz algebra

In this section, we construct a Lie 2-algebra from a Leibniz algebra via the skew-symmetrization. The notion of Leibniz algebras was introduced by Loday in [12], which is a vector space  $\mathfrak{g}$ , endowed with a linear map  $[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$  satisfying

$$[x, [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} = [[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [y, [x, z]_{\mathfrak{g}}]_{\mathfrak{g}}, \quad \forall x, y, z \in \mathfrak{g}. \quad (1)$$

The **left center** is given by

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y]_{\mathfrak{g}} = 0, \quad \forall y \in \mathfrak{g}\}. \quad (2)$$

It is obvious that  $Z(\mathfrak{g})$  is an ideal and the quotient Leibniz algebra  $\mathfrak{g}/Z(\mathfrak{g})$  is actually a Lie algebra since  $[x, x]_{\mathfrak{g}} \in Z(\mathfrak{g})$ , for all  $x \in \mathfrak{g}$ .

A Lie 2-algebra is a categorification of a Lie algebra, which is equivalent to a 2-term  $L_{\infty}$ -algebra (see [1], [14] for more details).

**Definition 1.1.** A Lie 2-algebra is a graded vector space  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$ , together with linear maps  $\{l_k : \wedge^k \mathfrak{g} \longrightarrow \mathfrak{g}, k = 1, 2, 3\}$  of degrees  $\deg(l_k) = k - 2$  satisfying the following equalities:

$$(a) \quad l_1 l_2(x, a) = l_2(x, l_1(a)),$$

$$(b) \quad l_2(l_1(a), b) = l_2(a, l_1(b)),$$

---

<sup>0</sup>Keyword: Leibniz algebras, omni-Lie algebras, representations, cohomologies

<sup>0</sup>MSC: 17B99, 55U15.

\*Research partially supported by NSF of China (11101179).

- (c)  $l_2(x, l_2(y, z)) + c.p. = l_1 l_3(x, y, z),$
- (d)  $l_2(x, l_2(y, a)) + l_2(y, l_2(a, x)) + l_2(a, l_2(x, y)) = l_3(x, y, l_1(a)),$
- (e)  $l_3(l_2(x, y), z, w) + c.p. = l_2(l_3(x, y, z), w) + c.p.,$

for all  $x, y, z, w \in \mathfrak{g}_0$ ,  $a, b \in \mathfrak{g}_1$ , where *c.p.* means cyclic permutations.

Given a Leibniz algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ , introduce the following skew-symmetric bracket on  $\mathfrak{g}$ :

$$[[x, y]] = \frac{1}{2}([x, y]_{\mathfrak{g}} - [y, x]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}, \quad (3)$$

and denote by  $J_{x,y,z}$  the corresponding Jacobiator, i.e.

$$J_{x,y,z} = [[x, [y, z]]] + [[y, [z, x]]] + [[z, [x, y]]]. \quad (4)$$

**Proposition 1.2.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$  be a Leibniz algebra.*

- (i) *For all  $x, y, z \in \mathfrak{g}$ , we have*

$$J_{x,y,z} = \frac{1}{4}([[[z, y]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [[x, z]_{\mathfrak{g}}, y]_{\mathfrak{g}} + [[y, x]_{\mathfrak{g}}, z]_{\mathfrak{g}}). \quad (5)$$

- (ii)  *$J_{x,y,z} \in Z(\mathfrak{g})$ , i.e.  $J_{x,y,z}$  is in the left center of  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ .*

- (iii) *For all  $x, y, z, w \in \mathfrak{g}$ , we have*

$$\begin{aligned} & [[x, J_{y,z,w}]] - [[y, J_{x,z,w}]] + [[z, J_{x,y,w}]] - [[w, J_{x,y,z}]] \\ & - J_{[[x,y]],z,w} + J_{[[x,z]],y,w} - J_{[[x,w]],y,z} - J_{[[y,z]],x,w} + J_{[[y,w]],x,z} - J_{[[z,w]],x,y} = 0. \end{aligned} \quad (6)$$

**Proof.** The first conclusion is obtained by straightforward computations. For any  $w \in \mathfrak{g}$ , by (1) and the fact that for all  $x \in \mathfrak{g}$ ,  $[x, x]_{\mathfrak{g}} \in Z(\mathfrak{g})$ , we have

$$\begin{aligned} [J_{x,y,z}, w]_{\mathfrak{g}} &= \frac{1}{4}([[[z, y]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [[x, z]_{\mathfrak{g}}, y]_{\mathfrak{g}} + [[y, x]_{\mathfrak{g}}, z]_{\mathfrak{g}}, w]_{\mathfrak{g}}) \\ &= \frac{1}{4}([[[z, [y, x]_{\mathfrak{g}}]_{\mathfrak{g}} - [y, [z, x]_{\mathfrak{g}}]_{\mathfrak{g}} - [[z, x]_{\mathfrak{g}}, y]_{\mathfrak{g}} + [[y, x]_{\mathfrak{g}}, z]_{\mathfrak{g}}, w]_{\mathfrak{g}}) \\ &= 0, \end{aligned}$$

which implies that  $J_{x,y,z} \in Z(\mathfrak{g})$ . At last, since the bracket  $[[\cdot, \cdot]]$  given by (4) is skew-symmetric, we have

$$\begin{aligned} & [[x, J_{y,z,w}]] - [[y, J_{x,z,w}]] + [[z, J_{x,y,w}]] - [[w, J_{x,y,z}]] \\ & - J_{[[x,y]],z,w} + J_{[[x,z]],y,w} - J_{[[x,w]],y,z} - J_{[[y,z]],x,w} + J_{[[y,w]],x,z} - J_{[[z,w]],x,y} \\ = & \underline{[[x, J_{y,z,w}]]} - \underline{[[y, J_{x,z,w}]]} + \underline{[[z, J_{x,y,w}]]} - \underline{[[w, J_{x,y,z}]]} \end{aligned}$$

$$\begin{aligned}
& - \llbracket \llbracket x, y \rrbracket, \llbracket z, w \rrbracket \rrbracket - \llbracket z, \llbracket w, \llbracket x, y \rrbracket \rrbracket \rrbracket - \llbracket w, \llbracket \llbracket x, y \rrbracket, z \rrbracket \rrbracket \\
& + \llbracket \llbracket x, z \rrbracket, \llbracket y, w \rrbracket \rrbracket + \llbracket y, \llbracket w, \llbracket x, z \rrbracket \rrbracket \rrbracket + \llbracket w, \llbracket \llbracket x, z \rrbracket, y \rrbracket \rrbracket \\
& - \llbracket \llbracket x, w \rrbracket, \llbracket y, z \rrbracket \rrbracket - \llbracket y, \llbracket z, \llbracket x, w \rrbracket \rrbracket \rrbracket - \llbracket z, \llbracket \llbracket x, w \rrbracket, y \rrbracket \rrbracket \\
& - \llbracket \llbracket y, z \rrbracket, \llbracket x, w \rrbracket \rrbracket - \llbracket x, \llbracket w, \llbracket y, z \rrbracket \rrbracket \rrbracket - \llbracket w, \llbracket \llbracket y, z \rrbracket, x \rrbracket \rrbracket \\
& + \llbracket \llbracket y, w \rrbracket, \llbracket x, z \rrbracket \rrbracket + \llbracket x, \llbracket z, \llbracket y, w \rrbracket \rrbracket \rrbracket + \llbracket z, \llbracket \llbracket y, w \rrbracket, x \rrbracket \rrbracket \\
& - \llbracket \llbracket z, w \rrbracket, \llbracket x, y \rrbracket \rrbracket - \llbracket x, \llbracket y, \llbracket z, w \rrbracket \rrbracket \rrbracket - \llbracket y, \llbracket \llbracket z, w \rrbracket, x \rrbracket \rrbracket \\
& = - \llbracket \llbracket x, y \rrbracket, \llbracket z, w \rrbracket \rrbracket + \llbracket \llbracket x, z \rrbracket, \llbracket y, w \rrbracket \rrbracket - \llbracket \llbracket x, w \rrbracket, \llbracket y, z \rrbracket \rrbracket \\
& - \llbracket \llbracket y, z \rrbracket, \llbracket x, w \rrbracket \rrbracket + \llbracket \llbracket y, w \rrbracket, \llbracket x, z \rrbracket \rrbracket - \llbracket \llbracket z, w \rrbracket, \llbracket x, y \rrbracket \rrbracket \\
& = 0.
\end{aligned}$$

The proof is finished. ■

Next, for a Leibniz algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ , we consider the graded vector space  $\mathcal{G} = Z(\mathfrak{g}) \oplus \mathfrak{g}$ , where  $Z(\mathfrak{g})$  is of degree 1,  $\mathfrak{g}$  is of degree 0. Define a degree  $-1$  differential  $l_1 = \mathbf{i} : Z(\mathfrak{g}) \rightarrow \mathfrak{g}$ , the inclusion. Define a degree 0 skew-symmetric bilinear map  $l_2$  and a degree 1 totally skew-symmetric trilinear map  $l_3$  on  $\mathcal{G}$  by

$$\begin{cases} l_2(x, y) = \llbracket x, y \rrbracket = \frac{1}{2}([x, y]_{\mathfrak{g}} - [y, x]_{\mathfrak{g}}) & \forall x, y \in \mathfrak{g}, \\ l_2(x, c) = -l_2(c, x) = \llbracket x, c \rrbracket = \frac{1}{2}[x, c]_{\mathfrak{g}} & \forall x \in \mathfrak{g}, c \in Z(\mathfrak{g}), \\ l_2(c_1, c_2) = 0 & \forall c_1, c_2 \in Z(\mathfrak{g}), \\ l_3(x, y, z) = J_{x, y, z} & \forall x, y, z \in \mathfrak{g}. \end{cases} \quad (7)$$

The following theorem is our main result in this section, which says that one can obtain a Lie 2-algebra via the skew-symmetrization of a Leibniz algebra.

**Theorem 1.3.** *With the above notations,  $(\mathcal{G}, l_1, l_2, l_3)$  is a Lie 2-algebra.*

**Proof.** By the definition of  $l_1$ ,  $l_2$  and  $l_3$ , it is obvious that Conditions (a)-(d) in Definition 1.1 hold. By (iii) in Proposition 1.2, Condition (e) also holds. Thus,  $(\mathcal{G}, l_1, l_2, l_3)$  is a Lie 2-algebra. ■

## 2 Representations of Leibniz algebras

The theory of representations and cohomologies of Leibniz algebras was introduced and studied in [13]. Especially, faithful representations and conformal representations of Leibniz algebras were studied in [3] and [9] respectively. See [4, 7] for more applications of cohomologies of Leibniz algebras.

**Definition 2.1.** *A representation of the Leibniz algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$  is a triple  $(V, l, r)$ , where  $V$  is a vector space equipped with two linear maps  $l : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  and  $r : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  such that the following equalities hold:*

$$l_{[x, y]_{\mathfrak{g}}} = [l_x, l_y], \quad r_{[x, y]_{\mathfrak{g}}} = [l_x, r_y], \quad r_y \circ l_x = -r_y \circ r_x, \quad \forall x, y \in \mathfrak{g}. \quad (8)$$

**Definition 2.2.** The Leibniz cohomology of  $\mathfrak{g}$  with coefficients in  $V$  is the cohomology of the cochain complex  $C^k(\mathfrak{g}, V) = \text{Hom}(\otimes^k \mathfrak{g}, V)$ , ( $k \geq 0$ ) with the coboundary operator

$$\partial : C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)$$

defined by

$$\begin{aligned} \partial c^k(x_1, \dots, x_{k+1}) &= \sum_{i=1}^k (-1)^{i+1} l_{x_i}(c^k(x_1, \dots, \widehat{x}_i, \dots, x_{k+1})) + (-1)^{k+1} r_{x_{k+1}}(c^k(x_1, \dots, x_k)) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \dots, \widehat{x}_i, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}). \end{aligned} \quad (9)$$

The resulting cohomology is denoted by  $H^\bullet(\mathfrak{g}; l, r)$ .

Obviously,  $(\mathbb{R}, 0, 0)$  is a representation of  $\mathfrak{g}$ , which is called the trivial representation. Denote the resulting cohomology by  $H^\bullet(\mathfrak{g})$ . Another important representation is the adjoint representation  $(\mathfrak{g}, \text{ad}_L, \text{ad}_R)$ , where  $\text{ad}_L : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$  and  $\text{ad}_R : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$  are defined as follows:

$$\text{ad}_L(x)(y) = [x, y]_{\mathfrak{g}}, \quad \text{ad}_R(x)(y) = [y, x]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \quad (10)$$

The resulting cohomology is denoted by  $H^\bullet(\mathfrak{g}; \text{ad}_L, \text{ad}_R)$ .

The graded vector space  $\bigoplus_k C^k(\mathfrak{g}, \mathfrak{g})$  equipped with the graded bracket

$$[\alpha, \beta] = \alpha \circ \beta + (-1)^{pq+1} \beta \circ \alpha, \quad \forall \alpha \in C^{p+1}(\mathfrak{g}, \mathfrak{g}), \beta \in C^{q+1}(\mathfrak{g}, \mathfrak{g}) \quad (11)$$

is a graded Lie algebra, where  $\alpha \circ \beta \in C^{p+q+1}(\mathfrak{g}, \mathfrak{g})$  is defined by

$$\begin{aligned} \alpha \circ \beta(x_1, \dots, x_{p+q+1}) &= \sum_{k=0}^p (-1)^{kq} \left( \sum_{\sigma \in sh(k, q)} \text{sgn}(\sigma) \alpha(x_{\sigma(1)}, \dots, x_{\sigma(k)}, \right. \\ &\quad \left. \beta(x_{\sigma(k+1)}, \dots, x_{\sigma(k+q)}, x_{k+q+2}, \dots, x_{p+q+1}) \right). \end{aligned}$$

See [2, 5] for more details. In particular, for  $\alpha \in C^2(\mathfrak{g}, \mathfrak{g})$ , we have

$$[\alpha, \alpha](x, y, z) = 2\alpha \circ \alpha(x, y, z) = 2 \left( \alpha(\alpha(x, y), z) - \alpha(x, \alpha(y, z)) + \alpha(y, \alpha(x, z)) \right). \quad (12)$$

Thus,  $\alpha$  defines a Leibniz algebra structure if and only if  $[\alpha, \alpha] = 0$ .

For a representation  $(V, l, r)$  of  $\mathfrak{g}$ , it is obvious that  $(V, l, 0)$  is a representation of  $\mathfrak{g}$  too. Thus, we have two semidirect product Leibniz algebras  $\mathfrak{g} \ltimes_{(l, r)} V$  and  $\mathfrak{g} \ltimes_{(l, 0)} V$  with the brackets  $[\cdot, \cdot]_{(l, r)}$  and  $[\cdot, \cdot]_{(l, 0)}$  respectively:

$$\begin{aligned} [x + u, y + v]_{(l, r)} &= [x, y]_{\mathfrak{g}} + l_x v + r_y u, \\ [x + u, y + v]_{(l, 0)} &= [x, y]_{\mathfrak{g}} + l_x v. \end{aligned}$$

The right action  $r$  induces a linear map  $\bar{r} : (\mathfrak{g} \oplus V) \otimes (\mathfrak{g} \oplus V) \longrightarrow \mathfrak{g} \oplus V$  as follows:

$$\bar{r}(x + u, y + v) = r_y u.$$

The adjoint actions maps  $\text{ad}_L$  and  $\text{ad}_R$  on the Leibniz algebra  $\mathfrak{g} \ltimes_{(l, 0)} V$  are given by

$$\text{ad}_L(x + u)(y + v) = [x, y]_{\mathfrak{g}} + l_x v, \quad \text{ad}_R(x + u)(y + v) = [y, x]_{\mathfrak{g}} + l_y u.$$

**Proposition 2.3.** *With the above notations,  $\bar{r}$  satisfies the following Maurer-Cartan equation on the Leibniz algebra  $\mathfrak{g} \ltimes_{(l,0)} V$ :*

$$\partial\bar{r} - \frac{1}{2}[\bar{r}, \bar{r}] = 0,$$

where  $\partial$  is the coboundary operator for the adjoint representation of  $\mathfrak{g} \ltimes_{(l,0)} V$ . Consequently, the Leibniz algebra  $\mathfrak{g} \ltimes_{(l,r)} V$  is a deformation of the Leibniz algebra  $\mathfrak{g} \ltimes_{(l,0)} V$  via the Maurer-Cartan element  $\bar{r}$ .

**Proof.** By direct computation, we have

$$\begin{aligned} \partial\bar{r}(x+u, y+v, z+w) &= \text{ad}_L(x+u)\bar{r}(y+v, z+w) - \text{ad}_L(y+v)\bar{r}(x+u, z+w) \\ &\quad - \text{ad}_R(z+w)\bar{r}(x+u, y+v) - \bar{r}([x+u, y+v]_{(l,0)}, z+w) \\ &\quad + \bar{r}(x+u, [y+v, z+w]_{(l,0)}) - \bar{r}(y+v, [x+u, z+w]_{(l,0)}) \\ &= l_x r_z v - l_y r_z u - r_z l_x v + r_{[y,z]_{\mathfrak{g}}} u - r_{[x,z]_{\mathfrak{g}}} v. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [\bar{r}, \bar{r}](x+u, y+v, z+w) &= 2(\bar{r}(\bar{r}(x+u, y+v), z+w) - \bar{r}(x+u, \bar{r}(y+v, z+w)) \\ &\quad + \bar{r}(y+v, \bar{r}(x+u, z+w))) \\ &= 2r_z r_y u. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left(\partial\bar{r} - \frac{1}{2}[\bar{r}, \bar{r}]\right)(x+u, y+v, z+w) &= l_x r_z v - l_y r_z u - r_z l_x v + r_{[y,z]_{\mathfrak{g}}} u - r_{[x,z]_{\mathfrak{g}}} v - r_z r_y u \\ &= l_x r_z v - l_y r_z u - r_z l_x v + r_{[y,z]_{\mathfrak{g}}} u - r_{[x,z]_{\mathfrak{g}}} v + r_z l_y u \\ &= 0. \end{aligned}$$

The proof is finished. ■

Let  $(V^*, l^*, 0)$  be the dual representation of the representation  $(V, l, 0)$  for  $\mathfrak{g}$ , where  $l^* : \mathfrak{g} \rightarrow \mathfrak{gl}(V^*)$  is given by

$$\langle l_x^*(\xi), u \rangle = -\langle \xi, l_x u \rangle, \quad \forall x \in \mathfrak{g}, \xi \in V^*, u \in V.$$

It is straightforward to see that  $(V^* \otimes V, l^* \otimes 1 + 1 \otimes l, 0)$  is also a representation of  $\mathfrak{g}$ , where  $l^* \otimes 1 + 1 \otimes l : \mathfrak{g} \rightarrow \mathfrak{gl}(V^* \otimes V)$  is given by

$$(l^* \otimes 1 + 1 \otimes l)_x(\xi \otimes u) = (l_x^* \xi) \otimes u + \xi \otimes l_x u, \quad \forall \xi \in V^*, u \in V.$$

Since  $V^* \otimes V \cong \mathfrak{gl}(V)$ , an element  $\xi \otimes u$  in  $V^* \otimes V$  can be identified with a linear map  $A \in \mathfrak{gl}(V)$  via  $A(v) = \langle \xi, v \rangle u$ .

**Proposition 2.4.** *With the above notations, for all  $A \in V^* \otimes V \cong \mathfrak{gl}(V)$ , we have*

$$(l^* \otimes 1 + 1 \otimes l)_x A = [l_x, A].$$

Moreover, the right action  $r : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is a 1-cocycle for the representation  $(V^* \otimes V, l^* \otimes 1 + 1 \otimes l, 0)$  of  $\mathfrak{g}$ .

**Proof.** Write  $A = \xi \otimes u$ , then we have

$$\begin{aligned} (l^* \otimes 1 + 1 \otimes l)_x A(v) &= (l^* \otimes 1 + 1 \otimes l)_x (\xi \otimes u)(v) = \left( (l_x^* \xi) \otimes u + \xi \otimes l_x u \right)(v) \\ &= \langle l_x^* \xi, v \rangle u + \langle \xi, v \rangle l_x u = -\langle \xi, l_x v \rangle u + \langle \xi, v \rangle l_x u \\ &= [l_x, A](v). \end{aligned}$$

Therefore, we have

$$\partial r(x, y) = (l^* \otimes 1 + 1 \otimes l)_x r(y) - r([x, y]_{\mathfrak{g}}) = [l_x, r_y] - r_{[x, y]_{\mathfrak{g}}} = 0,$$

which implies that  $r$  is a 1-cocycle. ■

### 3 Naive representations of Leibniz algebras

It is known that the aim of a representation is to realize an abstract algebraic structure as a class of linear transformations on a vector space. Such as a Lie algebra representation is a homomorphism from  $\mathfrak{g}$  to the general linear Lie algebra  $\mathfrak{gl}(V)$ . Unfortunately, the representation of a Leibniz algebra discussed above does not realize the abstract operation as a concrete linear operation. Therefore, it is reasonable for us to provide an alternative definition for the representation of Leibniz algebras. It is lucky that there is a god-given Leibniz algebra worked as a “general linear algebra” defined as follows:

Given a vector space  $V$ , then  $(V, l = \text{id}, r = 0)$  is a natural representation of  $\mathfrak{gl}(V)$ , which is viewed as a Leibniz algebra. The corresponding semidirect product Leibniz algebra structure on  $\mathfrak{gl}(V) \oplus V$  is given by

$$\{A + u, B + v\} = [A, B] + Av, \quad \forall A, B \in \mathfrak{gl}(V), \quad u, v \in V.$$

This Leibniz algebra is called **omni-Lie algebra** and denoted by  $\mathfrak{ol}(V)$ . The notion of an omni-Lie algebra was introduced by Weinstein in [15] as the linearization of a Courant algebroid. The notion of a Courant algebroid was introduced in [11], which has been widely applied in many fields both for mathematics and physics (see [6, 10] for more details). Its Leibniz algebra structure also played an important role when studying the integrability of Courant brackets [8].

Notice that the skew-symmetric bracket,

$$[[A + u, B + v]] = [A, B] + \frac{1}{2}(Av - Bu), \quad (13)$$

which is obtained via the skew-symmetrization of  $\{\cdot, \cdot\}$ , is used in his original definition. As a special case in Theorem 1.3,  $(\mathfrak{gl}(V) \oplus V, [[\cdot, \cdot]])$  is a Lie 2-algebra. Even though an omni-Lie algebra is not a Lie algebra, all Lie algebra structures on  $V$  can be characterized by the Dirac structures in  $\mathfrak{ol}(V)$ . In fact, the next proposition will show that every Leibniz algebra structure on  $V$  can be realized as a Leibniz subalgebra of  $\mathfrak{ol}(V)$ . For any  $\varphi : V \longrightarrow \mathfrak{gl}(V)$ , consider its graph

$$\mathcal{G}_\varphi = \{\varphi(u) + u \in \mathfrak{gl}(V) \oplus V \mid \forall u \in V\}.$$

**Proposition 3.1.** *With the above notations,  $\mathcal{G}_\varphi$  is a Leibniz subalgebra of  $\mathfrak{ol}(V)$  if and only if*

$$[\varphi(u), \varphi(v)] = \varphi(\varphi(u)v), \quad \forall u, v \in V. \quad (14)$$

Furthermore, under this condition,  $(V, [\cdot, \cdot]_\varphi)$  is a Leibniz algebra, where the linear map  $[\cdot, \cdot]_\varphi : V \otimes V \longrightarrow V$  is given by

$$[u, v]_\varphi = \varphi(u)v, \quad \forall u, v \in V. \quad (15)$$

**Proof.** Since  $\mathfrak{ol}(V)$  is a Leibniz algebra, we only need to show that  $\mathcal{G}_\varphi$  is closed if and only if (14) holds. The conclusion follows from

$$\{\varphi(u) + u, \varphi(v) + v\} = [\varphi(u), \varphi(v)] + \varphi(u)v.$$

The other conclusion is straightforward. The proof is finished. ■

Recall that a representation of a Lie algebra  $\mathfrak{g}$  on a vector space  $V$  is a Lie algebra homomorphism from  $\mathfrak{g}$  to the Lie algebra  $\mathfrak{gl}(V)$ , which realizes an abstract Lie algebra as a subalgebra of a concrete Lie algebra. Similarly, For a Leibniz algebra, we suggest the following definition:

**Definition 3.2.** *A naive representation of a Leibniz algebra  $\mathfrak{g}$  on a vector space  $V$  is a Leibniz algebra homomorphism  $\rho : \mathfrak{g} \longrightarrow \mathfrak{ol}(V)$ .*

According to the two components of  $\mathfrak{gl}(V) \oplus V$ , every linear map  $\rho : \mathfrak{g} \longrightarrow \mathfrak{ol}(V)$  can be split into two linear maps:  $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$  and  $\theta : \mathfrak{g} \longrightarrow V$ . Then, we have

**Proposition 3.3.** *A linear map  $\rho = \phi + \theta : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \oplus V$  is a naive representation of  $\mathfrak{g}$  if and only if  $(V, \phi, 0)$  is a representation of  $\mathfrak{g}$  and  $\theta : \mathfrak{g} \longrightarrow V$  is a 1-cocycle for the representation  $(V, \phi, 0)$ .*

**Proof.** On one hand, we have

$$\rho([x, y]_{\mathfrak{g}}) = \phi([x, y]_{\mathfrak{g}}) + \theta([x, y]_{\mathfrak{g}}).$$

On the other hand, we have

$$\{\rho(x), \rho(y)\} = \{\phi(x) + \theta(x), \phi(y) + \theta(y)\} = [\phi(x), \phi(y)] + \phi(x)\theta(y).$$

Thus,  $\rho$  is a homomorphism if and only if

$$\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)], \quad (16)$$

$$\theta([x, y]_{\mathfrak{g}}) = \phi(x)\theta(y). \quad (17)$$

By definition, Equalities (16) and (17) are equivalent to that  $(V, \phi, 0)$  is a representation and  $\theta : \mathfrak{g} \longrightarrow V$  is a 1-cocycle respectively. ■

**Theorem 3.4.** *Let  $(V, l, r)$  be a representation of the Leibniz algebra  $\mathfrak{g}$ . Then*

$$\rho = (l^* \otimes 1 + 1 \otimes l) + r : \mathfrak{g} \longrightarrow \mathfrak{ol}(V^* \otimes V)$$

*is a naive representation of  $\mathfrak{g}$  on  $V^* \otimes V$ .*

**Proof.** By Proposition 2.4,  $r : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$  is a 1-cocycle on  $\mathfrak{g}$  with the coefficients in the representation  $(V^* \otimes V, l^* \otimes 1 + 1 \otimes l, 0)$ . By Proposition 3.3,  $\rho = (l^* \otimes 1 + 1 \otimes l) + r$  is a homomorphism from  $\mathfrak{g}$  to  $\mathfrak{ol}(V^* \otimes V)$ . ■

A **trivial naive representation**  $\rho_T$  of  $\mathfrak{g}$  on  $\mathbb{R}$  is defined to be a homomorphism

$$\rho_T = \phi + \theta : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{R}) \oplus \mathbb{R}$$

such that  $\phi = 0$ . By Proposition 3.3, we have

**Proposition 3.5.** *Trivial naive representations of a Leibniz algebra are in one-to-one correspondence to  $\xi \in \mathfrak{g}^*$  such that  $\xi|_{[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}} = 0$ .*

The **adjoint naive representation**  $\mathfrak{ad}$  is defined to be the homomorphism

$$\mathfrak{ad} = \text{ad}_L + \text{id} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}.$$

## 4 Naive cohomology of Leibniz algebras

Let  $\rho : \mathfrak{g} \longrightarrow \mathfrak{ol}(V)$  be a naive representation of the Leibniz algebra  $\mathfrak{g}$ . It is obvious that  $\text{im}(\rho) \subset \mathfrak{ol}(V)$  is a Leibniz subalgebra so that one can define a set of  $k$ -cochains by

$$C^k(\mathfrak{g}; \rho) = \{f : \otimes^k \mathfrak{g} \longrightarrow \text{im}(\rho)\}$$

and an operator  $\delta : C^k(\mathfrak{g}; \rho) \longrightarrow C^{k+1}(\mathfrak{g}; \rho)$  by

$$\begin{aligned} \delta c^k(x_1, \dots, x_{k+1}) &= \sum_{i=1}^k (-1)^{i+1} \{\rho(x_i), c^k(x_1, \dots, \widehat{x}_i, \dots, x_{k+1})\} \\ &\quad + (-1)^{k+1} \{c^k(x_1, \dots, x_k), \rho(x_{k+1})\} \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \dots, \widehat{x}_i, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}). \end{aligned} \quad (18)$$

**Lemma 4.1.** *With the above notations, we have  $\delta^2 = 0$ .*

**Proof.** For all  $x \in \mathfrak{g}$  and  $u \in \text{im}(\rho)$ , define

$$l_x(u) = \{\rho(x), u\}, \quad r_x(u) = \{u, \rho(x)\}.$$

By the fact that  $\mathfrak{ol}(V)$  is a Leibniz algebra, we can deduce that  $(\text{im}(\rho); l, r)$  is a representation of  $\mathfrak{g}$  on  $\text{im}(\rho)$  in the sense of Definition 2.1.  $\delta$  is just the usual coboundary operator for this representation so that  $\delta^2 = 0$ . ■

Thus, we have a well-defined cochain complex  $(C^\bullet(\mathfrak{g}; \rho), \delta)$ . The resulting cohomology is called the **naive cohomology** and denoted by  $H_{naive}^\bullet(\mathfrak{g}; \rho)$ . In particular,  $H_{naive}^\bullet(\mathfrak{g})$  and  $H_{naive}^\bullet(\mathfrak{g}; \text{ad})$  denote the naive cohomologies corresponding to the trivial naive representation and adjoint naive representation of  $\mathfrak{g}$  respectively.

**Theorem 4.2.** *With the above notations, we have  $H_{naive}^\bullet(\mathfrak{g}) = H^\bullet(\mathfrak{g})$ .*

**Proof.** If  $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}} = \mathfrak{g}$ , there is only one trivial naive representation  $\rho = 0$  by Proposition 3.5. In this case, all the cochains are also 0. Thus,  $H_{naive}^\bullet(\mathfrak{g}) = 0$ . On the other hand, under the condition  $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}} = \mathfrak{g}$ , it is straightforward to deduce that for any  $\xi \in C^k(\mathfrak{g})$ ,  $\partial\xi = 0$  if and only if  $\xi = 0$ . Thus,  $H^\bullet(\mathfrak{g}) = 0$ .

If  $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}} \neq \mathfrak{g}$ , any  $0 \neq \xi \in \mathfrak{g}^*$  such that  $\xi|_{[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}} = 0$  gives rise to a trivial naive representation  $\rho_T$ . Furthermore, we have  $\text{im}(\rho_T) = \mathbb{R}$  and  $C^k(\mathfrak{g}) = \wedge^k \mathfrak{g}^*$ . Thus, the sets of cochains are the same associated to two kinds of representations. Since  $V$  is an abelian subalgebra in  $\mathfrak{ol}(V)$ , for any  $\xi \in C^k(\mathfrak{g})$ , we have

$$\begin{aligned} \delta\xi(x_1, \dots, x_{k+1}) &= \sum_{i=1}^k (-1)^{i+1} \{\rho_T(x_i), c^k(x_1, \dots, \widehat{x}_i, \dots, x_{k+1})\} \\ &\quad + (-1)^{k+1} \{c^k(x_1, \dots, x_k), \rho_T(x_{k+1})\} \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \dots, \widehat{x}_i, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \\ &= \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(x_1, \dots, \widehat{x}_i, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \\ &= \partial\xi(x_1, \dots, x_{k+1}). \end{aligned}$$



Thus, we have  $H_{naive}^\bullet(\mathfrak{g}) = H^\bullet(\mathfrak{g})$ . ■

For the adjoint naive representation  $\mathfrak{ad} = \text{ad}_L + \text{id} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}$ , any  $k$ -cochain  $f$  is uniquely determined by a linear map  $\mathfrak{f} : \otimes^k \mathfrak{g} \longrightarrow \mathfrak{g}$  such that

$$f = (\text{ad}_L \circ \mathfrak{f}, \mathfrak{f}) : \otimes^k \mathfrak{g} \longrightarrow \text{im}(\mathfrak{ad}). \quad (19)$$

**Theorem 4.3.** *With the above notations, we have  $H_{naive}^\bullet(\mathfrak{g}; \mathfrak{ad}) = H^\bullet(\mathfrak{g}; \text{ad}_L, \text{ad}_R)$ .*

**Proof.** Since any  $k$ -cochain  $f : \otimes^k \mathfrak{g} \longrightarrow \text{im}(\mathfrak{ad})$  is uniquely determined by a linear map  $\mathfrak{f} : \otimes^k \mathfrak{g} \longrightarrow \mathfrak{g}$  via (19). Thus, there is a one-to-one correspondence between the sets of cochains associated to the two kinds representations via  $f \longleftrightarrow \mathfrak{f}$ . Furthermore, we have

$$\begin{aligned} & \delta f(x_1, \dots, x_{k+1}) \\ = & \sum_{i=1}^k (-1)^{i+1} \{ \mathfrak{ad}(x_i), f(x_1, \dots, \widehat{x_i}, \dots, x_{k+1}) \} \\ & + (-1)^{k+1} \{ f(x_1, \dots, x_k), \mathfrak{ad}(x_{k+1}) \} \\ & + \sum_{1 \leq i < j \leq k+1} (-1)^i f(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \\ = & \sum_{i=1}^k (-1)^{i+1} \{ \text{ad}_L(x_i) + x_i, \text{ad}_L \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{k+1}) + \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{k+1}) \} \\ & + (-1)^{k+1} \{ \text{ad}_L \mathfrak{f}(x_1, \dots, x_k) + \mathfrak{f}(x_1, \dots, x_k), \text{ad}_L(x_{k+1}) + x_{k+1} \} \\ & + \sum_{1 \leq i < j \leq k+1} (-1)^i \left( \text{ad}_L \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \right. \\ & \left. + \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \right) \\ = & \sum_{i=1}^k (-1)^{i+1} \left( [\text{ad}_L(x_i), \text{ad}_L \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})] + \text{ad}_L(x_i) \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{k+1}) \right) \\ & + (-1)^{k+1} \left( [\text{ad}_L \mathfrak{f}(x_1, \dots, x_k), \text{ad}_L(x_{k+1})] + \text{ad}_L \mathfrak{f}(x_1, \dots, x_k) x_{k+1} \right) \\ & + \sum_{1 \leq i < j \leq k+1} (-1)^i \left( \text{ad}_L \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \right. \\ & \left. + \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \right) \\ = & \sum_{i=1}^k (-1)^{i+1} \left( \text{ad}_L [x_i, \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})]_{\mathfrak{g}} + [x_i, \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{k+1})]_{\mathfrak{g}} \right) \\ & + (-1)^{k+1} \left( \text{ad}_L [\mathfrak{f}(x_1, \dots, x_k), x_{k+1}]_{\mathfrak{g}} + [\mathfrak{f}(x_1, \dots, x_k), x_{k+1}]_{\mathfrak{g}} \right) \\ & + \sum_{1 \leq i < j \leq k+1} (-1)^i \left( \text{ad}_L \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \right. \\ & \left. + \mathfrak{f}(x_1, \dots, \widehat{x_i}, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_{k+1}) \right) \\ = & \text{ad}_L \partial \mathfrak{f}(x_1, \dots, x_{k+1}) + \partial \mathfrak{f}(x_1, \dots, x_{k+1}). \end{aligned}$$

Thus, we have  $\delta f = 0$  if and only if  $\partial \mathfrak{f} = 0$ . Similarly, we can prove that  $f$  is exact if and only if  $\mathfrak{f}$  is exact. Thus, the corresponding cohomologies are isomorphic. ■

At last, we consider a naive representation  $\rho$  such that the image of  $\rho$  is contained in the graph  $\mathcal{G}_\varphi$  for some linear map  $\varphi : V \longrightarrow \mathfrak{gl}(V)$  satisfying Eq. (14). In this case,  $\rho$  is of the form  $\rho = \varphi \circ \theta + \theta$ , where  $\theta : \mathfrak{g} \longrightarrow V$  is a linear map. A  $k$ -cochain  $f : \otimes^k \mathfrak{g} \longrightarrow \text{im}(\rho)$  is of the form  $f = \varphi \circ \mathfrak{f} + \mathfrak{f}$ , where  $\mathfrak{f} : \otimes^k \mathfrak{g} \longrightarrow V$  is a linear map.

Define left and right actions in the sense of Definition 2.1 by

$$l_x u = \text{pr}\{\rho(x), \varphi(u) + u\} = \varphi(\theta(x))u; \quad (20)$$

$$r_x u = \text{pr}\{\varphi(u) + u, \rho(x)\} = \varphi(u)\theta(x), \quad (21)$$

where  $\text{pr}$  is the projection from  $\mathfrak{gl}(V) \oplus V$  to  $V$ . Similar to Theorem 4.3, it is easy to prove that

**Theorem 4.4.** *Let  $\rho$  be a naive representation such that the image of  $\rho$  is contained in the graph  $\mathcal{G}_\varphi$  for some linear map  $\varphi : V \longrightarrow \mathfrak{gl}(V)$  satisfying Eq. (14). Then we have*

$$H_{naive}^\bullet(\mathfrak{g}; \rho) = H^\bullet(\mathfrak{g}; l, r),$$

where  $l$  and  $r$  are given by (20) and (21) respectively.

## References

- [1] J. Baez and A. S. Crans, Higher-Dimensional Algebra VI: Lie 2-Algebras, *Theory and Appl. Categ.* 12 (2004), 492-528.
- [2] D. Balavoine, Deformation of algebras over a quadratic operad, *Contemporary Maths.* AMS, 202 (1997) 207-234.
- [3] D. W. Barnes, Faithful representations of Leibniz algebras. *Proc. Amer. Math. Soc.* 141 (2013), no. 9, 2991-2995.
- [4] A. Fialowski, L. Magnin and A. Mandal, About Leibniz cohomology and deformations of Lie algebras. *J. Algebra*, 383 (2013) 63-77.
- [5] A. Fialowski and A. Mandal, Leibniz algebra deformations of a Lie algebra. *J. Math. Phys.* 49 (2008), no. 9, 093511, 11 pp.
- [6] M. Gualtieri, *Generalized Complex Geometry*, PhD thesis, St John's College, University of Oxford, Nov. 2003.
- [7] N. Hu, Y. Pei, and D. Liu, A cohomological characterization of Leibniz central extensions of Lie algebras. *Proc. Amer. Math. Soc.* 136 (2008), no. 2, 437-447.
- [8] M. K. Kinyon and A. Weinstein, Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces. *Amer. J. Math.* 123(2001):525-550,
- [9] P. S. Kolesnikov, Conformal representations of Leibniz algebras. (Russian) *Sibirsk. Mat. Zh.* 49 (2008), no. 3, 540-547; translation in *Sib. Math. J.* 49 (2008), no. 3, 429-435.
- [10] Y. Kosmann-Schwarzbach, Courant algebroids. A short history. *SIGMA Symmetry Integrability Geom. Methods Appl.* 9 (2013), Paper 014, 8 pp.

- [11] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Diff. Geom.* 45 (1997), 547-574.
- [12] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* (2), 39 (1993), 269-293.
- [13] J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.* 296 (1993), 139-158.
- [14] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, *J. Pure Appl. Alg.* 38 (1985), 313-322.
- [15] A. Weinstein, Omni-Lie algebras, Microlocal analysis of the Schrodinger equation and related topics (Japanese) (Kyoto, 1999). *Sūrikaiseikikenkyūsho Kōkyūroku* No. 1176 (2000), 95-102.